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Addendum

to the paper:

Finite Waiting Space Bulk Service System*

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5. Special Cases:

(I) $M/G^s/1/(N+1)$ System:

This is a "bulk service" system and the "service capacity" in this case is fixed. Thus we have

$$b_r = \begin{cases} 1 & \text{for} \quad r=0 \\ 0 & \text{for} \quad r \neq 0 \end{cases}$$

and hence $B_{s-i}=1=B_{s-i}(x)=B_s(x)$.

Therefore the expression (17) for $Q(x)$ reduces to

$$Q(x) = \frac{\sum_{i=0}^{s-1} p_i(x^s - x^i)}{x^s/K(x)-1}\,. \tag{18}$$

The expression corresponding to (6) for p_N is

$$p_N = \frac{\sum_{r=0}^{N-s-1} l_{N-r} p_{s+r} + l_N \left(\sum_{i=0}^{s} p_i\right)}{(1-l_s)} \tag{19}$$

(II) $E_s/G/1/(N+1)$ System:

In this case the inter-arrival time has the distribution

$$e^{-\lambda x} \frac{\lambda^s x^{s-1}}{(s-1)!}\, dx \tag{*}$$

Following Bhat [2] if we consider an input process which is Poisson with parameter λ, then the interval between the arrival points of consecutive s^{th} customers also has the distribution (*) above; and therefore instead of each customer of the system $E_s/G/1$ we can think of s hypothetical customers who arrive in a Poisson process and get served in a single batch. Consequently the study of $E_s/G/1$ system is identical with that of $M/G^s/1$ system and the results (18) and (19) above are valid in this case.

If Q'_n is the queue length just after the n^{th} departure in the system $E_s/G/1/(N+1)$ and Q_n be that in $M/G^s/1/(N+1)$, then Q'_n may be obtained from Q_n by using the relationship

$$Q'_n = \left[\frac{Q_n}{s}\right]$$

where $[\cdot]$ is the largest integer in the argument.

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(III) $M^r/G^s/1/(N+1)$ System:

This is a bulk queueing system, where arrivals are in groups of size r and service is in groups of size s. In this case the expression for k_j becomes

$$k_j = \begin{cases} \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^{j/r}}{(j/r)!} dG(t) & \text{for} \quad j = mr, \; m = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

The analysis of this system is similar to that of the $M/G^s/1/(N+1)$ system with minor changes in the transition probability matrix.

(IV) $M/G/1/(N+1)$ System:

In this case the service capacity s is one. The expression for $P(x)$ now becomes

$$P(x) = \frac{p_0(1-x)K(x)}{K(x)-x}. \tag{20}$$

The expression determining p_N is

$$p_N = \frac{\sum_{r=0}^{N} l_{N-r} p_{r+1} + l_N p_0}{1 - l_1}. \tag{21}$$

An Alternative Approach:

In this case the expressions for p_j's from (4) can be written explicitely as follows:

$$\begin{aligned} p_0 &= p_0 k_0 + p_1 k_0 \\ p_1 &= p_0 k_1 + p_1 k_1 + p_2 k_0 \\ p_2 &= p_0 k_2 + p_1 k_2 + p_2 k_1 + p_3 k_0 \\ p_3 &= p_0 k_3 + p_1 k_3 + p_2 k_2 + p_3 k_1 + p_4 k_0 \\ &\vdots \\ p_{n-1} &= p_0 k_{n-1} + p_1 k_{n-1} + p_2 k_{n-2} + \ldots + p_{n-1} k_1 + p_n k_0 , \end{aligned}$$

Note that we do not need the last equation for p_n, since the last column of the transition probability matrix depends on the previous ones. These equations can be solved resursively.

Let $\Lambda_j = p_j/p_0$, then since from the first equation $\Lambda_0 + \Lambda_1 = 1/k_0$, the solution is given by

$$\begin{aligned} \Lambda_0 &= 1 \\ \Lambda_1 k_0 &= (1 - k_0) \\ \Lambda_2 k_0 &= \Lambda_1 (1 - k_1) - k_1 \\ \Lambda_3 k_0 &= \Lambda_2 (1 - k_1) - \frac{k_2}{k_0} \\ \Lambda_4 k_0 &= \Lambda_3 (1 - k_1) - \Lambda_2 k_2 - \frac{k_3}{k_0} \\ \Lambda_5 k_0 &= \Lambda_4 (1 - k_1) - \Lambda_3 k_2 - \Lambda_2 k_3 - \frac{k_4}{k_0} \\ &\vdots \\ \Lambda_n k_0 &= \Lambda_{n-1}(1 - k_1) - \Lambda_{n-2} k_2 - \Lambda_{n-3} k_3 \ldots - \Lambda_2 k_{n-2} - \frac{k_{n-1}}{k_0}. \end{aligned}$$

Now the solution p_j satisfying (7) and $\Sigma\, p_j = 1$ is given by

$$p_j = \frac{A_j}{\sum_{i=0}^{N} A_i}, \quad j \leqq N .$$

(V) $M/M/1/(N+1)$ System:

In this case the service times have negative exponential distribution with parameter μ (say), i.e.,

$$g(t) = \mu e^{-\mu t}, \qquad t \geqq 0 \tag{22}$$

therefore

$$k_j = \left(\frac{\mu}{\lambda+\mu}\right)^j \left(\frac{\mu}{\lambda+\mu}\right) \tag{23}$$

Then following the arguments in Singh [8] we have

$$p_j = \begin{cases} \dfrac{(1-\rho)\rho^j}{1-\rho^{N+1}} & \rho < 1 \quad j = 0, 1, 2, \ldots, N \\[2ex] \dfrac{1}{N+1} & \rho = 1 \quad \text{where} \quad \rho = \dfrac{\lambda}{\mu} \end{cases} \tag{24}$$

In this case the average number of units in the system is

$$E(Q) = \begin{cases} \rho \left[\dfrac{1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)(1-\rho^{N+1})}\right], & \rho < 1 \\[2ex] \dfrac{N}{2} & \rho = 1 \end{cases} \tag{25}$$

The average time an arrival spends in the system is:

$$E(W) = \begin{cases} \dfrac{\rho}{\mu} \left[\dfrac{1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)(1-\rho^{N+1})}\right], & \rho < 1 \\[2ex] \dfrac{N}{2\mu} & \rho = 1 \end{cases} \tag{26}$$

(VI) $M/D/1/(N+1)$ System:

In this case

$$g(t) = \delta\left(t - \frac{1}{\mu}\right) = \begin{cases} 1 \text{ for } t = \dfrac{1}{\mu} \\[2ex] 0 \text{ for } t \neq \dfrac{1}{\mu} \end{cases} \tag{27}$$

and

$$k_j = e^{-\rho} \frac{\rho^j}{j!} \text{ where } \rho = \frac{\lambda}{\mu}. \tag{28}$$

Using the same arguments as in Singh [8], we have

$$p_j = \left[\sum_{r=1}^{j} \frac{e^{-\rho r}(-\rho r)^{j-r}}{(j-r)!} - \sum_{r=1}^{j-1} \frac{e^{\rho r}(-\rho r)^{j-r-1}}{(j-r-1)!}\right] p_0 \tag{29}$$

where p_0 is given by the following expression

$$p_0 = \left| \sum_{j=0}^{N} \sum_{r=1}^{j} \frac{e^{\rho r}(-\rho r)^{j-r}}{(j-r)!} - \sum_{j=0}^{N} \sum_{r=1}^{j-1} \frac{e^{\rho r}(-\rho r)^{j-r-1}}{(j-r-1)!} \right|^{-1} .$$

In this case, the average number of units in the system is given by

$$E(Q) = \left| \sum_{j=1}^{N} \sum_{r=1}^{j} \frac{j\, e^{\rho r}(-\rho r)^{j-r}}{(j-r)!} - \sum_{j=1}^{N} \sum_{r=1}^{j-1} j \cdot \frac{e^{\rho r}(-\rho r)^{j-r-1}}{(j-r-1)!} \right| p_0 \, .$$

We remark that the evaluation of p_j's in principle is a straight forward problem of power series expansion, yet for service times other than negative exponential, involves heavy algebraic computations. This fact is quite obviously demonstrated by the expression for p_j's in the $M/D/1$ system.